International Journal of Scientific & Engineering Research, Volume 5, Issue 8, August-2014
ISSN 2229-5518

The Steady-State Solution of Serial Channel with Feedback, Balking and Reneging Connected with Non-Serial Queuing Processes

Satyabir Mehla, Dr. Man Singh, Dr. Gulshan Taneja

Abstract— In the present paper, we have studied the general queuing model having feedback, balking and reneging in serial queuing processes connected with non serial queuing channels with random order selection for service and such models are of common occurrence in the administrative setup. We have also obtained the mean queue length of the model when queue discipline is first come first order.

Index Terms— Balking, Difference-differential, Exponential service, Feedback, Poisson arrivals, Random selection, Reneging, Steady-State, Waiting space.

1 Introduction

O' Brien (1954), Jackson (1954) and Hunt (1955) studied the problems of serial queues in the steady state with Poisson assumptions. In these studies, it is assumed that the unit must go through each service channel without leaving the system. Barrer (1955) obtained the steady-state solution of a single channel queuing model having Poisson input, exponential holding time, random selection where impatient customers leave the service facility after a wait of certain time. Finch (1959) studied simple queues with customers at random for service at a number of service stations in series where the arrival from outside was considered at the initial stage. Feedback is permitted either from the terminal server or from each server of the series to the queue waiting for service at that stage by imposing an upper limit on the number of customers in the system at any time. Singh (1984) studied the problem of serial queues introducing the concept of reneging. Singh and Singh (1994) worked on the network of queuing processes with impatient customers. Punam, Singh and Ashok (2011) found the steady-state solution of serial queuing processes where feedback is not permitted.

In our present work, the steady-state solutions are obtained for serial queuing processes with feedback, balking and reneging connected with non serial queuing channels in which

(i) A customer may join any channel from outside and leave the system at any stage after getting service.
(ii) M serial queuing processes with feedback, balking and reneging connected with N non serial queuing channels.
(iii) Feedback is permitted from each channel to its previous channel in serial channels.
(iv) The customer may balk due to long queue at each service channel.
(v) The impatient customer leaves the service facility after wait of certain time.
(vi) The input process in serial channels depends upon queue size and Poisson arrivals are followed.
(vii) Exponential service times are followed.
(viii) The queue discipline is random selection for service
(ix) Waiting space is infinite.

The expressions for marginal probabilities and mean queue length have also been derived whenever the queue discipline is first come first served.

2. Formulation of the Model

The system consists of the serial queues $Q_j\left(j=1,2,3,....,M\right)$ and non-serial channels $Q_{1i}(i=1,2,3,....N)$ with respective servers $S_j\left(j=1,2,3,....,M\right)$ and $S_{1i}\left(i=1,2,3,....N\right)$.Customers demanding different types of service arrive from outside the system in Poisson stream with parameters $\lambda_j\left(j=1,2,3,......,M\right)$ and $\lambda_{1i}\left(i=1,2,3,...N\right)$ at $Q_j\left(j=1,2,3,....,M\right)$ and $Q_{1i}(i=1,2,3,....N)$ but the sight of long queue at $Q_j(\mathrm{j}=1,2,3,....\mathrm{M})$ may discourage the fresh customer from joining it and may decide not to enter the service channel at $Q_j(\mathrm{j}=1,2,3,....\mathrm{M})$. Then the Poisson input rate at $Q_j(\mathrm{j}=1,2,3,....\mathrm{M})$ would be $\frac{\lambda_j}{n_j+1}$ where n_j is the

- *Satyabir Mehla. Assoc. Prof. R.K.S.D.(P.G) College, Kaithal, Haryana, India. PH-+919416311354. E-mail: satyabirsinghmehla@gmail.com*
- *Dr. Man Singh. Prof.(Retd.), Department of Mathematics and Statistics, C.C.S.H.A.U. Hisar, Haryana, India.*
- *Dr. Gulshan Taneja.Prof. Department of Mathematics, M.D.U. Rohtak, Haryana, India.*

queue size of $Q_j(\mathrm{j}=1,2,3,\ldots\mathrm{M})$. Further, the impatient customer joining any service channel $Q_j\left(j=1,2,3,\ldots,M\right)$ may leave the queue without getting service after wait of certain time. Service time distributions for servers $S_j\left(j=1,2,3,\ldots,M\right)$ and $S_{1i}\left(i=1,2,3,\ldots N\right)$ are mutually independent negative exponential distribution with parameters $\mu_j\left(j=1,2,\ldots,M\right)$ and $\mu_{1i}\left(i=1,2,3,\ldots N\right)$ respectively. After the completion of service at S_j, the customer either leaves the system with probability p_j or joins the next channel with probability $\frac{q_j}{n_{j+1}+1}$ or join back the previous channel with probability $\frac{r_j}{n_{j-1}+1}$ such that $p_j+\frac{q_j}{n_{j+1}+1}+\frac{r_j}{n_{j-1}+1}=1 \quad \left(j=1,2,3,\ldots M-1\right)$ and after the completion of service at S_M the customer either leaves the system with probability p_M or join back the previous channel with probability $\frac{r_M}{n_{M-1}+1}$ or join any queue $Q_{1i}\left(i=1,2,3,\ldots N\right)$ with probability $q_{Mi}\left(i=1,2,3,\ldots N\right)$ such that $p_M+\frac{r_M}{n_{M-1}+1}+\sum_{i=1}^{N}q_{Mi}=1$. It is being mentioned here that $r_j=0$ for $j=1$ as there is no previous channel of the first channel.

The applications of such models are of common occurrence. For example, consider the administration of a particular district in a particular state at the level of district head quarter consisting of Block Development officer, Tehsildar, Sub-Divisional Magistrate, District Magistrate etc. These officers correspond to the servers of serial channels. Education Department, Health Department, Irrigation Department etc. connected with the last server of serial queue correspond to non-serial channels. The people meet the officers of the district in connection with their problems. It is also a common practice that officers call the customers (people) for hearing randomly. The senior officer may send any customer to his junior if some information regarding the customer's problem is lacking. Further District Magistrate may send the customers to different departments such as Education, Health, Irrigation etc if there problems are related to such departments. The customer after seeing long queues before any serial service channel may decide not to enter the queue. It generally happens that person becomes impatient after joining the queue and may leave the channel without getting service.

3. FORMULATION OF EQUATIONS:

Define: $P\left(n_1,n_2,n_3,\ldots n_{M-1},n_M,m_1,m_2,m_3,\ldots m_{N-1},m_N;t\right)$ = the probability that at time 't' there are n_j customers (which may balk, renege or leave the system after being serviced or join the next phase or join back the previous channel) waiting before $S_j\left(j=1,2,3,\ldots,M-1,M\right)$; m_i customers(which may leave the system after being serviced) waiting before the severs $S_{1i}\left(i=1,2,3,\ldots N\right)$.

We define the operators $T_i\cdot$, $T\cdot_i$, $T\cdot_{i,i+1}$, $T_{i-1}\cdot,\cdot_i$ to act upon the vectors $\tilde{n}=\left(n_1,n_2,n_3,\ldots n_M\right)$ or $\tilde{m}=\left(m_1,m_2,m_3,\ldots m_N\right)$ as fallows

$$T_i\cdot\left(\tilde{n}\right)=\left(n_1,n_2,n_3,\ldots,n_i-1,\ldots n_M\right)$$
$$T\cdot_i\left(\tilde{n}\right)=\left(n_1,n_2,n_3,\ldots,n_i+1,\ldots,n_M\right)$$
$$T\cdot_i,_{i+1}\cdot\left(\tilde{n}\right)=\left(n_1,n_2,n_3,\ldots,n_i+1,n_{i+1}-1,\ldots,n_M\right)$$
$$T_{i-1}\cdot,\cdot_i\left(\tilde{n}\right)=\left(n_1,n_2,n_3,\ldots,n_{i-1}-1,n_i+1\ldots,n_M\right)$$

Following the procedure given by Kelly [5], we write the difference – differential equations as

$$\frac{d\,P\left(\tilde{n},\tilde{m};t\right)}{dt}=-\left[\begin{array}{l}\sum_{i=1}^{M}\frac{\lambda_i}{n_i+1}+\sum_{i=1}^{M}\delta\left(n_i\right)\left(\mu_i+C_{in_i}\right)\\+\sum_{j=1}^{N}\lambda_{1j}+\sum_{j=1}^{N}\delta\left(m_j\right)\mu_{1j}\end{array}\right]P\left(\tilde{n},\tilde{m};t\right)$$
$$+\sum_{i=1}^{M}\frac{\lambda_i}{n_i}P\left(T_i\cdot\left(\tilde{n}\right),\tilde{m};t\right)+\sum_{i=1}^{M}\left(\mu_i p_i+C_{in_i+1}\right)P\left(T\cdot_i\left(\tilde{n}\right),\tilde{m};t\right)$$
$$+\sum_{i=1}^{M-1}\mu_i\frac{q_i}{n_{i+1}}P\left(T\cdot_i,_{i+1}\cdot\left(\tilde{n}\right),\tilde{m};t\right)+\sum_{i=1}^{M}\mu_i\frac{r_i}{n_{i-1}}P\left(T_{i-1}\cdot,\cdot_i\left(\tilde{n}\right),\tilde{m};t\right).$$
$$+\sum_{j=1}^{N}\mu_M q_{Mj}P\left(n_1,n_2,\ldots n_M+1,T_j\cdot\left(\tilde{m}\right);t\right)$$
$$+\sum_{j=1}^{N}\lambda_{1j}P\left(\tilde{n},T_j\cdot\left(\tilde{m}\right);t\right)+\sum_{j=1}^{N}\mu_{1j}P\left(\tilde{n},T\cdot_j\left(\tilde{m}\right);t\right) \qquad (1)$$

for $n_i\geq 0 \quad \left(i=1,2,3,\ldots,M\right)$, $m_j\geq 0\left(j=1,2,3,\ldots N\right)$;

where $\delta\left(x\right)=\begin{cases}1 & if \quad x\neq 0\\ 0 & if \quad x=0\end{cases}$

and $P\left(\tilde{n},\tilde{m};t\right)=\tilde{0}$ if any of the arguments in negative.

4. STEADY–STATE EQUATIONS

We write the following Steady–state equations of the queuing model by equating the time-derivates to zero in the equation (1)

$$\left[\sum_{i=1}^{M}\frac{\lambda_i}{n_i+1}+\sum_{i=1}^{M}\delta\left(n_i\right)\left(\mu_i+C_{in_i}\right)+\sum_{j=1}^{N}\lambda_{1j}+\sum_{j=1}^{N}\delta\left(m_j\right)\mu_{1j}\right]P\left(\tilde{n},\tilde{m}\right)$$
$$=\sum_{i=1}^{M}\frac{\lambda_i}{n_i}P\left(T_i\cdot\left(\tilde{n}\right),\tilde{m}\right)+\sum_{i=1}^{M}\left(\mu_i p_i+C_{in_i+1}\right)P\left(T\cdot_i\left(\tilde{n}\right),\tilde{m}\right)$$
$$+\sum_{i=1}^{M-1}\mu_i\frac{q_i}{n_{i+1}}P\left(T\cdot_i,_{i+1}\cdot\left(\tilde{n}\right),\tilde{m}\right)+\sum_{i=1}^{M}\mu_i\frac{r_i}{n_{i-1}}P\left(T_{i-1}\cdot,\cdot_i\left(\tilde{n}\right),\tilde{m}\right)$$

$$+\sum_{j=1}^{N}\mu_M q_{Mj} P\left(n_1,n_2,....n_M+1,T_j.(\tilde{m})\right)$$
$$+\sum_{j=1}^{N}\lambda_{1j} P\left(\tilde{n},T_j.(\tilde{m})\right)+\sum_{j=1}^{N}\mu_{1j} P\left(\tilde{n},T._j(\tilde{m})\right) \quad (2)$$

for $n_i \geq 0 \quad (i=1,2,3,....,M)$, $m_j \geq 0\left(j=1,2,3,...N\right)$;

5. Steady-State Solutions

The solutions of the Steady-State equations (2) can be verified to be

$$P(\tilde{n},\tilde{m})=P(\tilde{0},\tilde{0})\left(\frac{1}{n_1!}\frac{\left(\lambda_1+\frac{\mu_2 r_2\rho_2}{(n_2+1)\left(\mu_2+C_{2n_2+1}\right)}\right)^{n_1}}{\prod_{i=1}^{n_1}(\mu_1+C_{1i})}\right).$$
$$.\left(\frac{1}{n_2!}\frac{\left(\lambda_2+\frac{\mu_1 q_1\rho_1}{(n_1+1)\left(\mu_1+C_{1n_1+1}\right)}+\frac{\mu_3 r_3\rho_3}{(n_3+1)\left(\mu_3+C_{3n_3+1}\right)}\right)^{n_2}}{\prod_{i=1}^{n_2}(\mu_2+C_{2i})}\right).$$
$$.\left(\frac{1}{n_3!}\frac{\left(\lambda_3+\frac{\mu_2 q_2\rho_2}{(n_2+1)\left(\mu_2+C_{2n_2+1}\right)}+\frac{\mu_4 r_4\rho_4}{(n_4+1)\left(\mu_4+C_{4n_4+1}\right)}\right)^{n_3}}{\prod_{i=1}^{n_3}(\mu_3+C_{3i})}\right).....$$
$$.\left(\frac{1}{n_{M-1}!}\frac{\left(\lambda_{M-1}+\frac{\mu_{M-2} q_{M-2}\rho_{M-2}}{(n_{M-2}+1)\left(\mu_{M-2}+C_{M-2n_{M-2}+1}\right)}+\frac{\mu_M r_M\rho_M}{(n_M+1)\left(\mu_M+C_{Mn_M+1}\right)}\right)^{n_{M-1}}}{\prod_{i=1}^{n_{M-1}}(\mu_{M-1}+C_{M-1i})}\right).$$
$$.\left(\frac{1}{n_M!}\frac{\left(\lambda_M+\frac{\mu_{M-1} q_{M-1}\rho_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)}\right)^{n_M}}{\prod_{i=1}^{n_M}(\mu_M+C_{Mi})}\right).$$
$$.\left(\frac{\lambda_{11}+\frac{\mu_M q_{M1}\rho_M}{(n_M+1)\left(\mu_M+C_{Mn_M+1}\right)}}{\mu_{11}}\right)^{m_1}\left(\frac{\lambda_{12}+\frac{\mu_M q_{M2}\rho_M}{(n_M+1)\left(\mu_M+C_{Mn_M+1}\right)}}{\mu_{12}}\right)^{m_2}$$
$$......\left(\frac{\lambda_{1N}+\frac{\mu_M q_{MN}\rho_M}{(n_M+1)\left(\mu_M+C_{Mn_M+1}\right)}}{\mu_{1N}}\right)^{m_N} \quad (3)$$

$n_i \geq 0 \quad (i=1,2,3,....,M)$, $m_j \geq 0\left(j=1,2,3,...N\right)$

where

$$\rho_1=\lambda_1+\frac{\mu_2 r_2\rho_2}{(n_2+1)\left(\mu_2+C_{2n_2+1}\right)}$$
$$\rho_2=\left(\lambda_2+\frac{\mu_1 q_1\rho_1}{(n_1+1)\left(\mu_1+C_{1n_1+1}\right)}+\frac{\mu_3 r_3\rho_3}{(n_3+1)\left(\mu_3+C_{3n_3+1}\right)}\right)$$
$$\rho_3=\lambda_3+\frac{\mu_2 q_2\rho_2}{(n_2+1)\left(\mu_2+C_{2n_2+1}\right)}+\frac{\mu_4 r_4\rho_4}{(n_4+1)\left(\mu_4+C_{4n_4+1}\right)} \quad (4)$$

....
....
....

$$\rho_{M-1}=\lambda_{M-1}+\frac{\mu_{M-2} q_{M-2}\rho_{M-2}}{(n_{M-2}+1)\left(\mu_{M-2}+C_{M-2n_{M-2}+1}\right)}+\frac{\mu_M r_M\rho_M}{(n_M+1)\left(\mu_M+C_{Mn_M+1}\right)}$$
$$\rho_M=\lambda_M+\frac{\mu_{M-1} q_{M-1}\rho_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)}$$

Solving these (4) M-equations for ρ_M with the help of determinants, we get

$$\rho_M=\frac{\begin{pmatrix}\lambda_M\Delta_{M-1}+\frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)(\mu_{M-1}+C_{M-1n_{M-1}+1})}\lambda_{M-1}\Delta_{M-2}+\\ \frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)(\mu_{M-1}+C_{M-1n_{M-1}+1})}\cdot\frac{q_{M-2}\mu_{M-2}}{(n_{M-2}+1)(\mu_{M-2}+C_{M-2n_{M-2}+1})}\lambda_{M-2}\Delta_{M-3}+\ldots\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ +\frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)(\mu_{M-1}+C_{M-1n_{M-1}+1})}\cdot\frac{q_{M-2}\mu_{M-2}}{(n_{M-2}+1)(\mu_{M-2}+C_{M-2n_{M-2}+1})}\ldots\\ \cdot\frac{q_3\mu_3}{(n_3+1)(\mu_3+C_{3n_3+1})}\lambda_3\Delta_2\\ +\frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)(\mu_{M-1}+C_{M-1n_{M-1}+1})}\cdot\frac{q_{M-2}\mu_{M-2}}{(n_{M-2}+1)(\mu_{M-2}+C_{M-2n_{M-2}+1})}\ldots\\ \cdot\frac{q_3\mu_3}{(n_3+1)(\mu_3+C_{3n_3+1})}\cdot\frac{q_2\mu_2}{(n_2+1)(\mu_2+C_{2n_2+1})}\lambda_2\Delta_1\\ +\frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)(\mu_{M-1}+C_{M-1n_{M-1}+1})}\cdot\frac{q_{M-2}\mu_{M-2}}{(n_{M-2}+1)(\mu_{M-2}+C_{M-2n_{M-2}+1})}\ldots\\ \cdot\frac{q_3\mu_3}{(n_3+1)(\mu_3+C_{3n_3+1})}\cdot\frac{q_2\mu_2}{(n_2+1)(\mu_2+C_{2n_2+1})}\cdot\frac{q_1\mu_1}{(n_1+1)(\mu_1+C_{1n_1+1})}\lambda_1\end{pmatrix}}{\left(\Delta_{M-1}-\frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)(\mu_{M-1}+C_{M-1n_{M-1}+1})}\cdot\frac{r_M\mu_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}\Delta_{M-2}\right)} \quad (5)$$

where

$$\Delta_M=\Delta_{M-1}-\frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)(\mu_{M-1}+C_{M-1n_{M-1}+1})}\cdot\frac{r_M\mu_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}\Delta_{M-2}$$

Where

$\Delta_1=1$

$$\Delta_2=\begin{vmatrix}1 & -\frac{\frac{r_2\mu_2}{n_2+1}}{\mu_2+c_{2n_2+1}}\\ -\frac{\frac{q_1\mu_1}{n_1+1}}{\mu_1+c_{1n_1+1}} & 1\end{vmatrix}$$

$$\Delta_3=\begin{vmatrix}1 & -\frac{\frac{r_2\mu_2}{n_2+1}}{\mu_2+c_{2n_2+1}} & 0\\ -\frac{\frac{q_1\mu_1}{n_1+1}}{\mu_1+c_{1n_1+1}} & 1 & -\frac{\frac{r_3\mu_3}{n_3+1}}{\mu_3+c_{3n_3+1}}\\ 0 & -\frac{\frac{q_2\mu_2}{n_2+1}}{\mu_2+c_{2n_2+1}} & 1\end{vmatrix}$$

……………………………………………………………………………….
……………………………………………………………………………….
……………………………………………………………………………….

(6)

$$\Delta_M=\begin{vmatrix}1 & -\frac{\frac{r_2}{n_2+1}\mu_2}{\mu_2+C_{2n_2+1}} & 0 & 0 & -- & 0 & 0 & 0\\ -\frac{\frac{q_1}{n_1+1}\mu_1}{\mu_1+C_{1n_1+1}} & 1 & -\frac{\frac{r_3}{n_3+1}\mu_3}{\mu_3+C_{3n_3+1}} & 0 & -- & 0 & 0 & 0\\ 0 & -\frac{\frac{q_2}{n_2+1}\mu_2}{\mu_2+C_{2n_2+1}} & 1 & -\frac{\frac{r_4}{n_4+1}\mu_4}{\mu_4+C_{4n_4+1}} & -- & 0 & 0 & 0\\ - & - & - & - & -- & - & - & -\\ - & - & - & - & -- & - & - & -\\ 0 & 0 & 0 & 0 & -- & -\frac{\frac{q_{M-2}}{n_{M-2}+1}\mu_{M-2}}{\mu_{M-2}+C_{M-2n_{M-2}+1}} & 1 & -\frac{\frac{r_M}{n_M+1}\mu_M}{\mu_M+C_{Mn_M+1}}\\ 0 & 0 & 0 & 0 & -- & 0 & -\frac{\frac{q_{M-1}}{n_{M-1}+1}\mu_{M-1}}{\mu_{M-1}+C_{M-1n_{M-1}+1}} & 1\end{vmatrix}$$

Since ρ_M is obtained, so we can get ρ_{M-1} by putting the value of ρ_M in the last equation of (4), ρ_{M-2} by putting the values of ρ_{M-1} and ρ_M in the last but one equation of (4). Continuing in this way, we shall obtain $\rho_{M-3},\rho_{M-4},---,\rho_3,\rho_2$ and ρ_1.

Thus, we write (3) as under

$$p(\tilde{n},\tilde{m})=P(\tilde{0},\tilde{0})\left(\frac{1}{n_1!}\frac{(\rho_1)^{n_1}}{\prod_{i=1}^{n_1}(\mu_1+C_{1i})}\right)\left(\frac{1}{n_2!}\frac{(\rho_2)^{n_2}}{\prod_{i=1}^{n_2}(\mu_2+C_{2i})}\right)\left(\frac{1}{n_3!}\frac{(\rho_3)^{n_3}}{\prod_{i=1}^{n_3}(\mu_3+C_{3i})}\right)\cdots$$

$$\left(\frac{1}{n_{M-1}!}\frac{(\rho_{M-1})^{n_{M-1}}}{\prod_{i=1}^{n_{M-1}}(\mu_{M-1}+C_{M-1i})}\right)\cdot\left(\frac{1}{n_M!}\frac{(\rho_M)^{n_M}}{\prod_{i=1}^{n_M}(\mu_M+C_{Mi})}\right)\left(\frac{\lambda_{11}+\frac{\mu_M q_{M1}\rho_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}}{\mu_{11}}\right)^{m_1}$$

$$\cdot\left(\frac{\lambda_{12}+\frac{\mu_M q_{M2}\rho_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}}{\mu_{12}}\right)^{m_2}\cdots\cdots\left(\frac{\lambda_{1N}+\frac{\mu_M q_{MN}\rho_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}}{\mu_{1N}}\right)^{m_N} \quad (7)$$

$n_i \geq 0 \quad (i=1,2,3,....,M)$, $m_j \geq 0 \, (j=1,2,3,...N)$

We obtain $P(\tilde{0},\tilde{0})$ from the normalizing conditions.

$$\sum_{\tilde{n}=\tilde{0},\tilde{m}=\tilde{0}}^{\infty} P(\tilde{n},\tilde{m})=1 \quad (8)$$

and with the restriction that traffic intensity of each service channel of the system is less than unity, C_{in_i} is the reneging rate at which customer renege after a wait of time T_{0i} whenever there are n_i customer in the service channel Q_i.

$$C_{in_i}=\frac{\mu_{1i}e^{-\frac{\mu_{1i}T_{0i}}{n_i}}}{1-e^{-\frac{\mu_{1i}T_{0i}}{n_i}}} \quad (i=1,2,3,....M)$$

Here it is mentioned that the customers leave the system at constant rate as long as there is a line, provided that the customers are served in the order in which they arrive. Putting $C_{in_i}=C_i \quad (i=1,2,3,...M)$ in the steady-state solution (3) then $\rho_i \, (i=1,2,3,.....M)$ will change accordingly and the steady-state solution reduces to

$$P(\tilde{n},\tilde{m})=P(\tilde{0},\tilde{0})\left(\frac{1}{n_1!}\left(\frac{\rho_1}{\mu_1+C_1}\right)^{n_1}\right)\left(\frac{1}{n_2!}\left(\frac{\rho_2}{\mu_2+C_2}\right)^{n_2}\right).$$

$$.\left(\frac{1}{n_3!}\left(\frac{\rho_3}{\mu_3+C_3}\right)^{n_3}\right).$$

$$\ldots\left(\frac{1}{n_{M-1}!}\left(\frac{\rho_{M-1}}{\mu_{M-1}+C_{M-1}}\right)^{n_{M-1}}\right)\left(\frac{1}{n_M!}\left(\frac{\rho_M}{\mu_M+C_M}\right)^{n_M}\right).$$

$$.\left(\frac{\lambda_{11}+\frac{\mu_M q_{M1}\rho_M}{(n_M+1)(\mu_M+C_M)}}{\mu_{11}}\right)^{m_1}\left(\frac{\lambda_{12}+\frac{\mu_M q_{M2}\rho_M}{(n_M+1)(\mu_M+C_M)}}{\mu_{12}}\right)^{m_2}$$

$$\cdots\cdots\left(\frac{\lambda_{1N}+\frac{\mu_M q_{MN}\rho_M}{(n_M+1)(\mu_M+C_M)}}{\mu_{1N}}\right)^{m_N} \quad (9)$$

We obtain $P(\tilde{0},\tilde{0})$ from (8) and (9) as

$$\left(P(\tilde{0},\tilde{0})\right)^{-1}=\prod_{i=1}^{M}e^{\frac{\rho_i}{\mu_i+C_i}}\prod_{j=1}^{N}\frac{1}{1-\rho_{1j}}$$

Where $\rho_{1j}=\dfrac{\lambda_{1j}+\dfrac{\mu_M q_{Mj}\rho_M}{(n_M+1)(\mu_M+C_M)}}{\mu_{1j}}, \quad j=1,2,3,...N$

Thus $P(\tilde{n},\tilde{m})$ is completely determined.

6. Steady-State Marginal Probabilities

Let $P(n_1)$ be the steady-state marginal probability that there are n_1 units in the queue before the first server. This is determined as

$$P(n_1)=\sum_{n_2,n_3,....n_{M=0}}^{\infty}\sum_{\tilde{m}=0}^{\infty}P(\tilde{n},\tilde{m})$$

$$=\sum_{n_2,n_3,....n_M=0}^{\infty}\sum_{\tilde{m}=\tilde{0}}^{\infty}P(\tilde{0},\tilde{0})\left(\frac{1}{n_1!}\left(\frac{\rho_1}{\mu_1+C_1}\right)^{n_1}\right)\left(\frac{1}{n_2!}\left(\frac{\rho_2}{\mu_2+C_2}\right)^{n_2}\right).$$

$$.\left(\frac{1}{n_3!}\left(\frac{\rho_3}{\mu_3+C_3}\right)^{n_3}\right)\ldots\left(\frac{1}{n_{M-1}!}\left(\frac{\rho_{M-1}}{\mu_{M-1}+C_{M-1}}\right)^{n_{M-1}}\right)\left(\frac{1}{n_M!}\left(\frac{\rho_M}{\mu_M+C_M}\right)^{n_M}\right).$$

$$.(\rho_{11})^{m_1}(\rho_{12})^{m_2}\ldots(\rho_{1N})^{m_N}$$

Thus $P(n_1)=\dfrac{1}{n_1!}\left(\dfrac{\rho_1}{\mu_1+C_1}\right)^{n_1} e^{-(\frac{\rho_1}{\mu_1+C_1})} \qquad n_1>0$

Similarly

$$P(n_2)=\frac{1}{n_2!}\left(\frac{\rho_2}{\mu_2+C_2}\right)^{n_2} e^{-(\frac{\rho_2}{\mu_2+C_2})} \qquad n_2>0$$

$$P(n_M) = \frac{1}{n_M!}\left(\frac{\rho_M}{\mu_M + C_M}\right)^{n_M} e^{-(\frac{\rho_M}{\mu_M + C_M})} \qquad n_M > 0$$

Further let $P(m_1), P(m_2), P(m_3), ..., P(m_N)$ be the steady-state marginal probabilities that there are $m_1, m_2, m_3, ..., m_N$ customers waiting before server $S_{1i} (i = 1, 2, 3, ..., N)$ respectively.

$$P(m_1) = \sum_{\tilde{n}=0}^{\infty} \sum_{m_2, m_3, ... m_N = 0}^{\infty} P(\tilde{n}, \tilde{m})$$

$$= \sum_{\tilde{n}=0}^{\infty} \sum_{m_2, m_3, ... m_N = 0}^{\infty} P(\tilde{0}, \tilde{0}) \left(\frac{1}{n_1!}\left(\frac{\rho_1}{\mu_1 + C_1}\right)^{n_1}\right)\left(\frac{1}{n_2!}\left(\frac{\rho_2}{\mu_2 + C_2}\right)^{n_2}\right).$$

$$.\left(\frac{1}{n_3!}\left(\frac{\rho_3}{\mu_3 + C_3}\right)^{n_3}\right)....\left(\frac{1}{n_{M-1}!}\left(\frac{\rho_{M-1}}{\mu_{M-1} + C_{M-1}}\right)^{n_{M-1}}\right)\left(\frac{1}{n_M!}\left(\frac{\rho_M}{\mu_M + C_M}\right)^{n_M}\right).$$

$$.(\rho_{11})^{m_1} (\rho_{12})^{m_2} ...(\rho_{1N})^{m_N}$$

$$= (\rho_{11})^{m_1} (1 - \rho_{11}) \qquad m_1 > 0$$

Similarly

$$P(m_2) = (\rho_{12})^{m_2} (1 - \rho_{12}) \qquad m_2 > 0$$

....

$$P(m_N) = (\rho_{1N})^{m_N} (1 - \rho_{1N}) \qquad m_N > 0$$

7. Mean Queue Length

Mean queue length before the server S_1 is determined by

$$L_1 = \sum_{n_1=0}^{\infty} n_1 P(n_1) = \sum_{n_1=0}^{\infty} n_1 \frac{1}{n_1!}\left(\frac{\rho_1}{\mu_1 + C_1}\right)^{n_1} e^{-(\frac{\rho_1}{\mu_1 + C_1})}$$

$$= \frac{\rho_1}{\mu_1 + C_1}$$

Similarly

$$L_2 = \frac{\rho_2}{\mu_2 + C_2}$$

$$L_M = \frac{\rho_M}{\mu_M + C_M}$$

Mean queue length before the server S_{11} is determined as

$$L_{11} = \frac{\rho_{11}}{1 - \rho_{11}}$$

Similarly

$$L_{1j} = \frac{\rho_{1j}}{1 - \rho_{1j}} \qquad j = 2, 3, ..., N$$

Hence mean queue length of the system is

$$L = \sum_{k=1}^{M} L_k + \sum_{j=1}^{N} L_{1j}$$

Numerical Solutions of Mean Queue Length

Servers in Series S_M	Arrival rate λ_M before server S_M	Reneging rate C_M before server S_M	Service rate μ_M before server S_M	Marginal mean queue length before the server S_M $L_M=\rho_M/(\mu_M+C_M)$	Non-Serial Servers S_{1N}	Arrival rate λ_{1N} before servers S_{1N}	Prob. of joining the non-serial servers q_{M1N}	Service rate μ_{1N} before servers S_{1N}	Mean Queue Length before the servers $S_{1N}=\rho_{1N}/(1-\rho_{1N})$	Sum of Marginal Mean Queue Lengths of Serial and Non-Serial Servers
1	5	4	6	0.525890209	1	6	0.03	15	0.741019863	1.266910071
2	8	6	10	0.522451159	2	8	0.04	20	0.721816479	1.244267638
3	7	5	9	0.54003238	3	9	0.06	22	0.743867166	1.283899546
4	12	8	14	0.552314485	4	7	0.07	25	0.419193416	0.9715079
5	14	7	15	0.65252882	5	9	0.09	27	0.532728889	1.185257709
6	10	9	11	0.541496414	6	5	0.08	29	0.22799976	0.769496174
7	12	10	15	0.491879063	7	7	0.01	23	0.472882335	0.964761399
8	6	4	8	0.521399259	8	10	0.07	24	0.762678829	1.284078089
9	16	13	18	0.528765623	9	8	0.02	30	0.387883972	0.916649595
10	10	7	11	0.594011237	10	11	0.05	28	0.685159305	1.279170542

Mean queue length of the system = 11.16599866

REFERENCES

[1] D.Y.Barrer, "A Waiting line problem characterized by impatient customers and indifferent clerk,"*Journal of Operation Research Society of America*,1955,Vol. 3, 360.

[2] P.D.Finch, "Cyclic queues,"*Opns. Res. Quart*, 1959, Vol. 9, No. 1.

[3] R.R. P.Jackson, "Queuing system with phase type service,"*operat. Res. Quart*.,Vol.5, No. 2. pp. 109-120,1954.

[4] G.C. Hunt,"Sequential arrays of waiting lines,"*opps. Res.,*1955, Vol. 4, No. 674-683.

[5] F. P. Kelly, "Reversibility and stochastic networks," Wiley, New York, 1979.

[6] G.G.O'Brien,"The solution of some queuing problems,"*J.Soc. Ind. Appl. Math.*, 1954, Vol. 2, 132-142.

[7] M. Singh, "Steady-state behaviour of serial queuing processes with impatient customers,"*Math. operationsforsch, U.statist. Ser. Statist.,* 1984, Vol. 15.No 2, 289-298.

[8] Punam, M. Singh and Ashok, "The Steady-State solution of serial queuing processes with reneging due to long queue and urgent message," *Applied science periodical*, 2011, Vol. XII, No. 1.